Shen et al. *Journal of Inequalities and Applications* 2013, 2013:448
http://www.journalofinequalitiesandapplications.com/content/2013/1/448

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the strong convergence and some inequalities for negatively superadditive dependent sequences

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Abstract

In this paper, we study the Marcinkiewicz-type strong law of large numbers, Hajek-Renyi-type inequality and other inequalities for negatively superadditive dependent (NSD) sequences. As an application, the integrability of supremum for NSD random variables is obtained. Our results extend the corresponding ones of Christofides and Vaggelatou (J. Multivar. Anal. 88:138-151, 2004) and Liu *et al.* (Stat. Probab. Lett. 43:99-105, 1999).

MSC: 60F15

Keywords: negatively superadditive dependent; strong law of large numbers; system reliability; integrability of supremum

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. $S_n \doteq \sum_{i=1}^{n} X_i$, $n \geq 1$, $S_0 \doteq 0$. The concept of negatively associated (NA) random variables was introduced by Joag-Dev and Proschan [1]. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated if for every pair of disjoint subsets $A, B \subset \{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\big(f(X_i, i \in A), f(X_j, j \in B)\big) \leq 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

The concept of negatively superadditive dependent (NSD) random variables was introduced by Hu [2] based on the class of superadditive functions. Superadditive structure functions have important reliability interpretations, which describe whether a system is more series-like or more parallel-like [3].

Definition 1.1 (Kemperman [4]) A function $\phi : R^n \to R$ is called superadditive if $\phi(\mathbf{x} \vee \mathbf{y}) + \phi(\mathbf{x} \wedge \mathbf{y}) \geq \phi(\mathbf{x}) + \phi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in R^n$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.

Definition 1.2 (Hu [2]) A random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is said to be negatively superadditive dependent (NSD) if

$$E\phi(X_1, X_2, \dots, X_n) \le E\phi\left(X_1^*, X_2^*, \dots, X_n^*\right), \tag{1.1}$$

where $X_1^*, X_2^*, \dots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in (1.1) exist.

Hu [2] gave an example illustrating that NSD does not imply NA, and Hu posed an open problem whether NA implies NSD. Christofides and Vaggelatou [5] solved this open problem and indicated that NA implies NSD. Negatively superadditive dependent structure is an extension of negatively associated structure and sometimes more useful than negatively associated structure. Moreover, we can get many important probability inequalities for NSD random variables. For example, the structure function of a monotone coherent system can be superadditive [3], so inequalities derived from NSD can give one-side or two-side bounds of the system reliability. The notion of NSD random variables has wide applications in multivariate statistical analysis and reliability theory.

Eghbal *et al.* [6] derived two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables under the assumption that $\{X_i, i \ge 1\}$ is a sequence of nonnegative NSD random variables with $EX_i^r < \infty$ for all $i \ge 1$ and some $r > 1$. Eghbal *et al.* [7] provided some Kolmogorov inequality for quadratic forms $T_n = \sum_{1 \le i < j \le n} X_i X_j$ and weighted quadratic forms $Q_n = \sum_{1 \le i < j \le n} a_{ij} X_i X_j$, where $\{X_i, i \ge 1\}$ is a sequence of nonnegative NSD uniformly bounded random variables. Shen *et al.* [8] obtained the Khintchine-Kolmogorov convergence theorem and strong stability for NSD random variables. Strong convergence for NA sequences and other dependent sequences has been extensively investigated. For example, Sung [9, 10] obtained the complete convergence results for identically distributed NA random variables and ρ^*-mixing random variables respectively, Zhou *et al.* [11] studied complete convergence for identically distributed ρ^*-mixing random variables under a suitable moment condition, Zhou [12] discussed complete moment convergence of moving average process under φ-mixing assumptions.

This paper is organized as follows. In Section 2, some preliminary lemmas and inequalities for NSD random variables are provided. In Section 3, Marcinkiewicz-type strong law of large numbers, Hajek-Renyi-type inequalities and the integrability of supremum for NSD random variables are presented. These results extend the corresponding results of Christofides and Vaggelatou [5] and Liu *et al.* [13].

Throughout the paper, $X_1^*, X_2^*, \dots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i. C denotes a positive constant not depending on n, which may be different in various places. $a_n \ll b_n$ represents that there exists a constant $C > 0$ such that $a_n \le C b_n$ for all sufficiently large n.

2 Preliminaries

Lemma 2.1 (Hu [2]) *If $(X_1, X_2, \dots, X_n)$ is NSD, then $(X_{i_1}, X_{i_2}, \dots, X_{i_m})$ is NSD for any $1 \le i_1 < i_2 < \cdots < i_m$, $2 \le m < n$.*

Lemma 2.2 (Hu [2]) *Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be an NSD random vector, and let $\mathbf{X}^* = (X_1^*, X_2^*, \dots, X_n^*)$ be an independent vector such that X_i^* and X_i have the same distribution* *for each i. Then, for any nondecreasing convex function f and $n \geq 1$,*

$$Ef\left(\max_{1\leq k\leq n}\sum_{i=1}^{k}X_i\right) \leq Ef\left(\max_{1\leq k\leq n}\sum_{i=1}^{k}X_i^*\right). \tag{2.1}$$

Lemma 2.3 (Toeplitz lemma) *Let $\{a_n, n \geq 1\}$ be a sequence of real numbers. If $b_n = \sum_{k=1}^{n} a_k \uparrow \infty$ and $\lim_{n\to\infty} x_n = x$ (finite), then $\lim_{n\to\infty} \frac{1}{b_n}\sum_{k=1}^{n} a_k x_k = x$.*

Shen *et al.* [8] obtained the following two results. Lemma 2.4 comes from the proof of Theorem 2.1 of [8]. Lemma 2.5 is the Khintchine-Kolmogorov-type convergence theorem for NSD random variables.

Lemma 2.4 (Shen *et al.* [8]) *Let $X_1, X_2, \ldots, X_n$ be NSD random variables with mean zero and finite second moments. Then, for $n \geq 1$,*

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}X_i\right|^2\right) \leq 2\sum_{i=1}^{n}EX_i^2.$$

Lemma 2.5 (Shen *et al.* [8]) *Let $\{X_n, n \geq 1\}$ be a sequence of NSD random variables. Assume that*

$$\sum_{n=1}^{\infty}\operatorname{Var}X_n < \infty, \tag{2.2}$$

then $\sum_{n=1}^{\infty}(X_n - EX_n)$ converges almost surely.

3 Main results

Theorem 3.1 (Marcinkiewicz-type strong law of large numbers for NSD) *Let $\{X_n, n \geq 1\}$ be a sequence of NSD identically distributed random variables. There exists some $0 < p < 2$ such that*

$$E|X_1|^p < \infty. \tag{3.1}$$

If $1 \leq p < 2$, we further assume that $EX_1 = 0$. Then

$$\lim_{n\to\infty} n^{-\frac{1}{p}}\sum_{k=1}^{n}X_k = 0, \quad a.s. \tag{3.2}$$

Proof Let $Y_n = -n^{1/p}I(X_n < -n^{1/p}) + X_nI(|X_n| \leq n^{1/p}) + n^{1/p}I(X_n > n^{1/p})$. By (3.1),

$$\sum_{n=1}^{\infty}P(X_n \neq Y_n) = \sum_{n=1}^{\infty}P\big(|X_n| > n^{1/p}\big) = \sum_{n=1}^{\infty}P\big(|X_1| > n^{1/p}\big) \leq E|X_1|^p < \infty.$$

Therefore $P(X_n \neq Y_n, \text{i.o.}) = 0$ follows from the Borel-Cantelli lemma. Thus (3.2) is equivalent to the following:

$$\lim_{n\to\infty} n^{-\frac{1}{p}}\sum_{k=1}^{n}Y_k = 0, \quad \text{a.s.}$$

So, in order to prove (3.2), we need only to prove

$$\lim_{n\to\infty} n^{-\frac{1}{p}} \sum_{k=1}^{n} (Y_k - EY_k) = 0, \quad \text{a.s.}, \tag{3.3}$$

$$\lim_{n\to\infty} n^{-\frac{1}{p}} \sum_{k=1}^{n} EY_k = 0. \tag{3.4}$$

Firstly, we prove (3.3). According to Lemma 2.5, it suffices to prove

$$\sum_{n=1}^{\infty} \operatorname{Var}\left(\frac{Y_n}{n^{1/p}}\right) < \infty.$$

By $0 < p < 2$ and (3.1),

$$\begin{aligned}
\sum_{n=1}^{\infty} \operatorname{Var}\left(\frac{Y_n}{n^{1/p}}\right) &\le \sum_{n=1}^{\infty} \frac{EY_n^2}{n^{2/p}} \\
&= \sum_{n=1}^{\infty} n^{-2/p}\left[E\left(X_1^2 I\left(|X_1| \le n^{1/p}\right)\right) + E\left(n^{2/p} I\left(|X_1| > n^{1/p}\right)\right)\right] \\
&= \sum_{n=1}^{\infty} \left[n^{-2/p} E\left(X_1^2 I\left(|X_1| \le n^{1/p}\right)\right) + P\left(|X_1| > n^{1/p}\right)\right].
\end{aligned}$$

Notice that

$$\sum_{n=1}^{\infty} P\left(|X_1| > n^{1/p}\right) \le E|X_1|^p < \infty,$$

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{-2/p} EX_1^2 I\left(|X_1| \le n^{1/p}\right) &= \sum_{n=1}^{\infty} n^{-2/p} \sum_{k=1}^{n} EX_1^2 I\left(k-1 < |X_1|^p \le k\right) \\
&= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} n^{-2/p} EX_1^2 I\left(k-1 < |X_1|^p \le k\right) \\
&= \sum_{k=1}^{\infty} k^{-2/p} EX_1^2 I\left(k-1 < |X_1|^p \le k\right) \\
&\quad + \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} n^{-2/p} EX_1^2 I\left(k-1 < |X_1|^p \le k\right) \\
&\le \sum_{k=1}^{\infty} k^{-2/p} Ek^{2/p} I\left(k-1 < |X_1|^p \le k\right) \\
&\quad + \sum_{k=1}^{\infty} EX_1^2 I\left(k-1 < |X_1|^p \le k\right) \int_k^{\infty} x^{-2/p}\, dx \\
&\le \sum_{k=1}^{\infty} P\left(k-1 < |X_1|^p \le k\right) \\
&\quad + \sum_{k=1}^{\infty} \frac{p}{2-p} k^{\frac{p-2}{p}} E|X_1|^p k^{\frac{2-p}{p}} I\left(k-1 < |X_1|^p \le k\right)
\end{aligned}$$

$$\begin{aligned}&\ll \sum_{k=1}^{\infty} E|X_1|^p I\big(k-1<|X_1|^p\le k\big)\\&=E|X_1|^p<\infty.\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty}\operatorname{Var}\left(\frac{Y_n}{n^{1/p}}\right)<\infty.$$

Secondly, we prove (3.4).

(i) If $p=1$, then

$$|EY_n|\le \big|EX_1I\big(|X_1|\le n\big)\big|+E\big(nI\big(|X_1|>n\big)\big).$$

It is easy to see that

$$\begin{aligned}&\lim_{n\to\infty}EX_1I\big(|X_1|\le n\big)=EX_1=0,\\&\lim_{n\to\infty}E\big(nI\big(|X_1|>n\big)\big)\le\lim_{n\to\infty}E|X_1|I\big(|X_1|>n\big)=0.\end{aligned}$$

Therefore, it has $\lim_{n\to\infty}EY_n=0$. By Lemma 2.3, we can get (3.4) immediately.

(ii) If $0<p<1$, by (3.1)

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{|EY_n|}{n^{1/p}}&\le\sum_{n=1}^{\infty}n^{-1/p}\big[E|X_n|I\big(|X_n|\le n^{1/p}\big)+E\big(n^{1/p}I\big(|X_n|>n^{1/p}\big)\big)\big]\\&=\sum_{n=1}^{\infty}\big[n^{-1/p}E|X_1|I\big(|X_1|\le n^{1/p}\big)+EI\big(|X_1|>n^{1/p}\big)\big].\end{aligned}$$

Notice that

$$\begin{aligned}&\sum_{n=1}^{\infty}EI\big(|X_1|>n^{1/p}\big)=\sum_{n=1}^{\infty}P\big(|X_1|>n^{1/p}\big)\le E|X_1|^p<\infty,\\&\sum_{n=1}^{\infty}n^{-1/p}E|X_1|I\big(|X_1|\le n^{1/p}\big)=\sum_{n=1}^{\infty}\sum_{j=1}^{n}n^{-1/p}E|X_1|I\big(j-1<|X_1|^p\le j\big)\\&\qquad=\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}n^{-1/p}E|X_1|I\big(j-1<|X_1|^p\le j\big)\\&\qquad=\sum_{j=1}^{\infty}j^{-1/p}E|X_1|I\big(j-1<|X_1|^p\le j\big)\\&\qquad\quad+\sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty}n^{-1/p}E|X_1|I\big(j-1<|X_1|^p\le j\big)\\&\qquad\le\sum_{j=1}^{\infty}j^{-1/p}Ej^{1/p}I\big(j-1<|X_1|^p\le j\big)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{\infty} E|X_1| I\big(j-1<|X_1|^p \le j\big) \int_j^{\infty} x^{-1/p}\,dx \\
&\ll \sum_{j=1}^{\infty} E|X_1|^p I\big(j-1<|X_1|^p \le j\big) \\
&= E|X_1|^p < \infty,
\end{aligned}
$$

we can get (3.4) from Kronecker's lemma.

(iii) If $1<p<2$, by $EX_1=0$ and (3.1),

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{|EY_n|}{n^{1/p}} &\le \sum_{n=1}^{\infty} n^{-1/p}\big[\big|EX_1 I\big(|X_1|>n^{1/p}\big)\big| + E\big(n^{1/p} I\big(|X_1|>n^{1/p}\big)\big)\big] \\
&\le \sum_{n=1}^{\infty}\big[n^{-1/p} E|X_1| I\big(|X_1|>n^{1/p}\big) + P\big(|X_1|>n^{1/p}\big)\big].
\end{aligned}
$$

Notice that

$$
\sum_{n=1}^{\infty} P\big(|X_1|>n^{1/p}\big) \le E|X_1|^p < \infty,
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{-1/p} E|X_1| I\big(|X_1|>n^{1/p}\big) &= \sum_{n=1}^{\infty}\sum_{j=n}^{\infty} n^{-1/p} E|X_1| I\big(j<|X_1|^p \le j+1\big) \\
&= \sum_{j=1}^{\infty}\sum_{n=1}^{j} n^{-1/p} E|X_1| I\big(j<|X_1|^p \le j+1\big) \\
&\le \sum_{j=1}^{\infty} E|X_1|^p |X_1|^{1-p} I\big(j<|X_1|^p \le j+1\big) \int_0^j x^{-1/p}\,dx \\
&\ll \sum_{j=1}^{\infty} E|X_1|^p I\big(j<|X_1|^p \le j+1\big) \\
&\le \sum_{j=0}^{\infty} E|X_1|^p I\big(j<|X_1|^p \le j+1\big) \\
&= E|X_1|^p < \infty.
\end{aligned}
$$

Thus we can also get (3.4) from Kronecker's lemma. □

Theorem 3.2 *Let $\{X_n, n\ge 1\}$ be a sequence of NSD random variables with mean zero and finite second moments. Let $\{b_n, n\ge 1\}$ be a sequence of positive nondecreasing real numbers. Then, for any $\varepsilon>0$ and $n\ge 1$,*

$$
P\left(\max_{1\le k\le n}\left|\frac{1}{b_k}\sum_{j=1}^{k} X_j\right| > \varepsilon\right) \le \frac{8}{\varepsilon^2}\sum_{j=1}^{n}\frac{EX_j^2}{b_j^2}. \tag{3.5}
$$

Proof Without loss of generality, we may assume that $b_n \ge 1$ for all $n\ge 1$. Let $\alpha=\sqrt{2}$. For $i\ge 0$, define $A_i=\{1\le k\le n: \alpha^i \le b_k < \alpha^{i+1}\}$. Then A_i may be an empty set. For $A_i \ne \emptyset$, we let $\nu(i) = \max\{k : k \in A_i\}$ and t_n be the index of the last nonempty set A_i. Obviously, $A_iA_j = \emptyset$ if $i \neq j$ and $\sum_{i=0}^{t_n} A_i = \{1, 2, \ldots, n\}$. It is easy to see that $\alpha_i \le b_k \le b_{\nu(i)} < \alpha^{i+1}$ if $k \in A_i$. By Markov's inequality and Lemma 2.4, we have

$$\begin{aligned}
P\left\{\max_{1\le k\le n}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| > \varepsilon\right\} &= P\left\{\max_{0\le i\le t_n, A_i\neq\emptyset}\max_{k\in A_i}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| > \varepsilon\right\} \\
&= P\left\{\bigcup_{0\le i\le t_n, A_i\neq\emptyset}\left(\max_{k\in A_i}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| > \varepsilon\right)\right\} \\
&\le \sum_{i=0, A_i\neq\emptyset}^{t_n} P\left\{\max_{k\in A_i}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| > \varepsilon\right\} \\
&\le \sum_{i=0, A_i\neq\emptyset}^{t_n} P\left\{\frac{1}{\alpha^i}\max_{1\le k\le \nu(i)}\left|\sum_{j=1}^{k}X_j\right| > \varepsilon\right\} \\
&\le \frac{1}{\varepsilon^2}\sum_{i=0, A_i\neq\emptyset}^{t_n}\frac{1}{\alpha^{2i}}E\left(\max_{1\le k\le \nu(i)}\left|\sum_{j=1}^{k}X_j\right|^2\right) \\
&\le \frac{2}{\varepsilon^2}\sum_{i=0, A_i\neq\emptyset}^{t_n}\frac{1}{\alpha^{2i}}\sum_{j=1}^{\nu(i)}EX_j^2 \\
&= \frac{2}{\varepsilon^2}\sum_{j=1}^{n}EX_j^2\sum_{i=0, A_i\neq\emptyset, \nu(i)\ge j}^{t_n}\frac{1}{\alpha^{2i}}.
\end{aligned} \tag{3.6}$$

Now we estimate $\sum_{i=0, A_i\neq\emptyset, \nu(i)\ge j}^{t_n}\frac{1}{\alpha^{2i}}$. Let $i_0 = \min\{i : A_i \neq \emptyset, \nu(i) \ge j\}$. Then $b_j \le b_{\nu(i_0)} < \alpha^{i_0+1}$ follows from the definition of $\nu(i)$. Therefore

$$\begin{aligned}
\sum_{i=0, A_i\neq\emptyset, \nu(i)\ge j}^{t_n}\frac{1}{\alpha^{2i}} &< \sum_{i=i_0}^{\infty}\frac{1}{\alpha^{2i}} = \frac{1}{1-\frac{1}{\alpha^2}}\frac{1}{\alpha^{2i_0}} \\
&= \frac{\alpha^2}{1-\frac{1}{\alpha^2}}\frac{1}{\alpha^{2(i_0+1)}} < \frac{\alpha^2}{1-\frac{1}{\alpha^2}}\frac{1}{b_j^2} = \frac{4}{b_j^2}.
\end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7), we obtain (3.5) immediately. □

Theorem 3.3 *Let $\{X_n, n \ge 1\}$ be a sequence of NSD random variables with mean zero and finite second moments. Let $\{b_n, n \ge 1\}$ be a sequence of positive nondecreasing real numbers. Then, for any $\varepsilon > 0$ and for any positive integer $m < n$,*

$$\begin{aligned}
P\left(\max_{m\le k\le n}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| > \varepsilon\right) &\le \frac{4}{\varepsilon^2 b_m^2}\left\{\sum_{j=1}^{m}EX_j^2 + 2\sum_{1\le k<j\le m}\operatorname{Cov}(X_k, X_j)\right\} \\
&\quad + \frac{32}{\varepsilon^2}\sum_{j=m+1}^{n}\frac{EX_j^2}{b_j^2}.
\end{aligned} \tag{3.8}$$

Proof Observe that

$$\max_{m\le k\le n}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| \le \left|\frac{1}{b_m}\sum_{j=1}^{m}X_j\right| + \max_{m+1\le k\le n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}X_j\right|.$$

Then

$$\begin{aligned} P\left\{\max_{m\le k\le n}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right|>\varepsilon\right\} &\le P\left\{\left|\frac{1}{b_m}\sum_{j=1}^{m}X_j\right|>\frac{\varepsilon}{2}\right\}+P\left\{\max_{m+1\le k\le n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}X_j\right|>\frac{\varepsilon}{2}\right\} \\ &\doteq I+II. \end{aligned} \tag{3.9}$$

For I, by Markov's inequality, we have

$$\begin{aligned} I &\le \frac{4}{\varepsilon^2 b_m^2}E\left|\sum_{j=1}^{m}X_j\right|^2 = \frac{4}{\varepsilon^2 b_m^2}\left[\sum_{j=1}^{m}EX_j^2+2\sum_{1\le k<j\le m}EX_kX_j\right] \\ &= \frac{4}{\varepsilon^2 b_m^2}\left[\sum_{j=1}^{m}EX_j^2+2\sum_{1\le k<j\le m}\operatorname{Cov}(X_k,X_j)\right]. \end{aligned} \tag{3.10}$$

For II, we will apply Theorem 3.2 to $\{X_{m+i}, 1 \le i \le n-m\}$ and $\{b_{m+i}, 1 \le i \le n-m\}$. According to Lemma 2.1, $\{X_{m+i}, 1 \le i \le n-m\}$ is NSD. Noting that

$$\max_{m+1\le k\le n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}X_j\right| = \max_{1\le k\le n-m}\left|\frac{1}{b_{m+k}}\sum_{j=1}^{k}X_{m+j}\right|,$$

thus, by Theorem 3.2, we obtain

$$\begin{aligned} II &= P\left\{\max_{m+1\le k\le n}\left|\frac{1}{b_k}\sum_{j=m+1}^{k}X_j\right|>\frac{\varepsilon}{2}\right\} \\ &= P\left\{\max_{1\le k\le n-m}\left|\frac{1}{b_{m+k}}\sum_{j=1}^{k}X_{m+j}\right|>\frac{\varepsilon}{2}\right\} \\ &\le \frac{8}{(\frac{\varepsilon}{2})^2}\sum_{j=1}^{n-m}\frac{EX_{m+j}^2}{b_{m+j}^2} \\ &= \frac{32}{\varepsilon^2}\sum_{j=m+1}^{n}\frac{EX_j^2}{b_j^2}. \end{aligned} \tag{3.11}$$

Therefore the desired result (3.8) follows from (3.9)-(3.11) immediately. □

Theorem 3.4 *Let $\{b_n, n \ge 1\}$ be a sequence of positive nondecreasing real numbers. Let $\{X_n, n \ge 1\}$ be a sequence of NSD random variables with mean zero and $\sum_{j=1}^{\infty}\frac{EX_j^2}{b_j^2} < \infty$. If $0 < r < 2$, then*

$$E\left(\sup_{n\ge1}\left|\frac{S_n}{b_n}\right|^r\right) \le 1+\frac{8r}{2-r}\sum_{j=1}^{\infty}\frac{EX_j^2}{b_j^2} < \infty. \tag{3.12}$$

Proof For $\forall 0 < r < 2$, $t > 0$, by Theorem 3.2, it follows that

$$\begin{aligned} P\left(\sup_{n\geq 1}\left|\frac{S_n}{b_n}\right|^r > t\right) &= P\left[\bigcup_{N=1}^{\infty}\left(\max_{1\leq n\leq N}\left|\frac{S_n}{b_n}\right| > t^{1/r}\right)\right] \\ &= \lim_{N\to\infty} P\left(\max_{1\leq n\leq N}\left|\frac{S_n}{b_n}\right| > t^{1/r}\right) \\ &\leq \lim_{N\to\infty}\frac{8}{t^{2/r}}\sum_{j=1}^{N}\frac{EX_j^2}{b_j^2} = \frac{8}{t^{2/r}}\sum_{j=1}^{\infty}\frac{EX_j^2}{b_j^2}. \end{aligned}$$

Thus

$$\begin{aligned} E\left(\sup_{n\geq 1}\left|\frac{S_n}{b_n}\right|^r\right) &= \int_0^\infty P\left(\sup_{n\geq 1}\left|\frac{S_n}{b_n}\right|^r > t\right) dt \\ &= \int_0^1 P\left(\sup_{n\geq 1}\left|\frac{S_n}{b_n}\right|^r > t\right) dt + \int_1^\infty P\left(\sup_{n\geq 1}\left|\frac{S_n}{b_n}\right|^r > t\right) dt \\ &\leq 1 + 8\sum_{j=1}^{\infty}\frac{EX_j^2}{b_j^2}\int_1^\infty t^{-2/r}\, dt = 1 + \frac{8r}{2-r}\sum_{j=1}^{\infty}\frac{EX_j^2}{b_j^2} < \infty. \end{aligned}$$

□

Example 3.5 Similar to the proof of Theorem 2.1 of Shen *et al.* [8], we can get the following inequalities for NSD random variables.

Let $p \geq 1$. Suppose that $\{X_n, n \geq 1\}$ is a sequence of NSD random variables with mean zero and $E|X_n|^p < \infty$. Then, for $n \geq 1$,

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq 2E\left|\sum_{i=1}^{n} X_i^*\right|^p, \tag{3.13}$$

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq 2^{3-p}\sum_{i=1}^{n} E|X_i|^p \quad \text{for } 1 \leq p \leq 2, \tag{3.14}$$

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq C_p\left\{\left(\sum_{i=1}^{n} EX_i^2\right)^{p/2} + \sum_{i=1}^{n} E|X_i|^p\right\} \quad \text{for } p > 2, \tag{3.15}$$

where C_p is a positive constant depending only on p.

In fact, taking $f(x) = [\max(0, x)]^p$ in (2.1), we can get (3.13) similar to the proof of Theorem 2.1 in [8]. Following the same line of arguments as in the proof of Theorem 2 of Shao [14], we can obtain (3.14) and (3.15) immediately.

Equation (3.15) provides the Rosenthal-type inequality for NSD random variables, which is one of the most interesting inequalities in probability theory. Hu [2] pointed out that the Rosenthal-type inequality remains true for NSD random variables. Here we give the proof of this inequality and provide other two inequalities.

Remark 3.6 Theorem 3.1 provides Marcinkiewicz-type strong law of large numbers for NSD random variables. Marcinkiewicz strong law of large numbers for independent sequences and other dependent sequences was studied by many authors. See, for example, Lin *et al.* [15] for independent sequences, Wu and Jiang [16] for $\tilde{\rho}$-mixing sequences, Wu

[17] for PNQD sequences, Wang *et al.* [18] for Martingale difference sequences, and so forth.

Remark 3.7 If $\{X_n, n \geq 1\}$ is a sequence of NA random variables with finite second moments, Chrisofides and Vaggelatou [5] obtained the following result: for any $\varepsilon > 0$,

$$P\left(\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}(X_j - EX_j)\right| > \varepsilon\right) \leq \frac{32}{\varepsilon^2 b_m^2}\sum_{j=1}^{m}\operatorname{Var} X_j + \frac{32}{\varepsilon^2}\sum_{j=m+1}^{n}\frac{\operatorname{Var} X_j}{b_j^2}.$$

So, if we further assume that $EX_n = 0$, $n \geq 1$, then

$$P\left(\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| > \varepsilon\right) \leq \frac{32}{\varepsilon^2 b_m^2}\sum_{j=1}^{m}EX_j^2 + \frac{32}{\varepsilon^2}\sum_{j=m+1}^{n}\frac{EX_j^2}{b_j^2}.$$

If $\{X_n, n \geq 1\}$ is a sequence of NA, then $\{X_n, n \geq 1\}$ is a sequence of NSD, by $\operatorname{Cov}(X_k, X_j) \leq 0$ and (3.8)

$$P\left(\max_{m\leq k\leq n}\left|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right| > \varepsilon\right) \leq \frac{4}{\varepsilon^2 b_m^2}\sum_{j=1}^{m}EX_j^2 + \frac{32}{\varepsilon^2}\sum_{j=m+1}^{n}\frac{EX_j^2}{b_j^2}.$$

Hence Theorem 3.2 and Theorem 3.3 extend the Hajek-Renyi-type inequalities for NA random variables (Chrisofides and Vaggelatou [5]) and the factors are improved. As an application of Theorem 3.2, Theorem 3.4 extends the result of NA random variables (Liu *et al.* [13]) to NSD random variables.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Acknowledgements
Article is supported by the National Natural Science Foundation of China (11171001, 11201001), Doctoral Research Start-up Funds Project of Anhui University, Key Program of Research and Development Foundation of Hefei University (13KY05ZD) and Natural Science Foundation of Anhui Province (1208085QA03).

Received: 25 July 2013 Accepted: 30 August 2013 Published: 06 Nov 2013

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10.1186/1029-242X-2013-448
Cite this article as: Shen et al.: **On the strong convergence and some inequalities for negatively superadditive dependent sequences.** *Journal of Inequalities and Applications* 2013, 2013:448